[2] Liepmann, H. W. and Skinner, G. T., "Shearing Stress Measurements by Use of Heated Element," TN 3268, 1954, NACA.

[3] Bellhouse, B. J. and Schultz, D. L., "The Measurement of Skin Friction in Supersonic Flow by Means of Heated Thin Film Gauges," R and M 3490, 1968, Aeronautical Research Council.

[4] Owen, F. K., "Transition Experiments on a Flat Plate at Subsonic and Supersonic Speeds," *AIAA Journal*, Vol. 8, No. 3, March 1970, pp. 518–523.

Radiant Heat Transfer at the Vertex of Adjoint Plates

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Introduction

The Fredholm integral equations of the second kind that govern radiant heat transfer do not lend themselves to analytical treatment. As a result there are few exact solutions. Moreover, when these equations are singular, certain difficulties are encountered in obtaining numerical solutions. A good example is that of radiant heat transfer for adjoint plates, which was first studied by Sparrow et al.[1] and more recently by Love and Turner.[2] Because the governing integral equation is singular at the vertex of the adjoint plates, the numerical solutions encounter convergence difficulties in this region. It appears, moreover, that these authors could not obtain satisfactory convergence when the angle between the adjoint plates was small. This note is concerned with the analytical treatment at the vertex, which is of some theoretical as well as practical interest. The exact value of the heat transfer at the vertex is obtained together with the asymptotic behavior near the vertex.

Heat Transfer at the Vertex

The adjoint-plate configuration is shown in Fig. 1. The upper and lower plates have the same length, temperature, and surface properties (grey and diffuse), and they are separated by a vacuum. If the nondimensional distance and radiosity are denoted by $X = x/L$ and $\beta(X) = R(X)/\epsilon\sigma T^4$, the governing equation for $\beta(X)$ appears as[1,2]

$$\beta(X) = 1 + \lambda \int_0^1 \beta(Y)K(X,Y)dY \tag{1}$$

where

$$\lambda \equiv \tfrac{1}{2}\rho \sin^2\theta \tag{2}$$

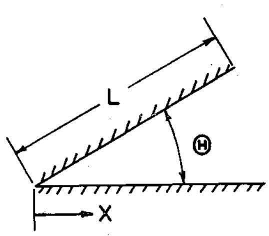

Fig. 1 Adjoint-plate configuration.

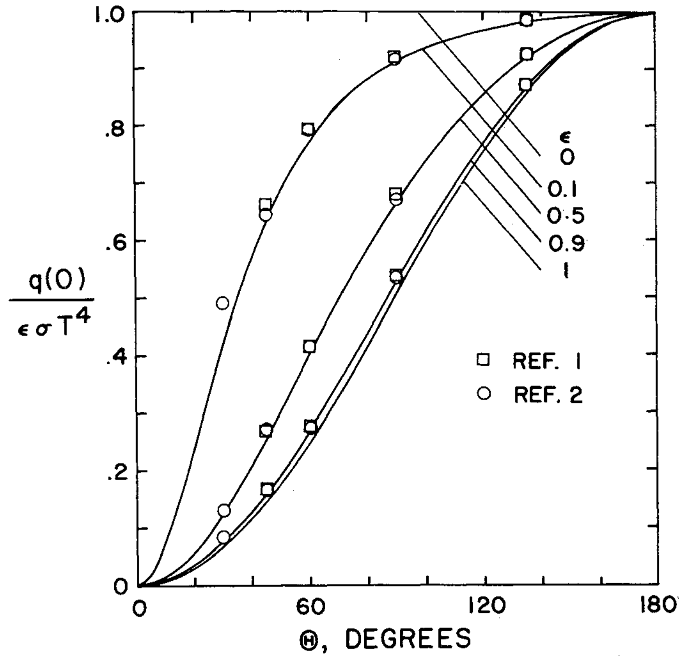

Fig. 2 Heat transfer at vertex.

$$K(X,Y) = XY[X^2 - 2XY\cos\theta + Y^2]^{-3/2} \tag{3}$$

and $\rho = (1 - \epsilon)$ is the reflectance.

Equation (1) can be written alternatively by means of the substitution $Y = Xu$;

$$\beta(X) = 1 + \lambda \int_0^{1/X} \beta(Xu)K(1,u)du \tag{4}$$

The radiosity at the vertex is determined by setting $X = 0$ in Eq. (4). We obtain

$$\beta(0) = 1 + \lambda\beta(0) \int_0^{\infty} K(1,u)du \tag{5}$$

After evaluating the integral in Eq. (5) by means of standard integral tables, we solve for $\beta(0)$ and obtain

$$\beta(0) = [1 - \rho\cos^2(\theta/2)]^{-1} \tag{6}$$

This is the exact value. The heat transfer is determined from the radiosity[1,2] and is found to be

$$\frac{q(0)}{\epsilon\sigma T^4} = \frac{1 - \epsilon\beta(0)}{\rho} = \frac{\sin^2(\theta/2)}{1 - \rho\cos^2(\theta/2)} \tag{7}$$

Equation (7) is plotted in Fig. 2 for various values of the emittance $\epsilon = (1 - \rho)$. Also shown are the numerical values obtained by Sparrow et al.[1] and Love and Turner.[2] The numerical values are seen to fall slightly above the exact values for the smaller values of θ and ϵ. This discrepancy arises because of the singular behavior of Eq. (1) at $X = 0$, which causes loss of accuracy in the numerical solution in this region. An asymptotic investigation reveals the nonanalytic behavior of the solution near $X = 0$.

Asymptotic Behavior

Preliminary investigation of the solution of Eq. (1) by means of the Neumann series[3] leads to the assumed behavior

$$\beta(X) \sim \beta_0 + \beta_1 X^{1-\alpha},\ X \to 0,\ 0 < \alpha < 1 \tag{8}$$

The constants β_0, β_1, and α are to be determined by substituting Eq. (8) into Eq. (4) and collecting like powers of X asymptotically as $X \to 0$. This procedure yields

$$\beta_0 + \beta_1 X^{1-\alpha} = (1 + \lambda\beta_0 b) - \lambda[\beta_0 + (\beta_1/\alpha)]X + \lambda\beta_1 Q(\alpha,\theta)X^{1-\alpha} + 0(X^2),\ X \to 0 \tag{9}$$

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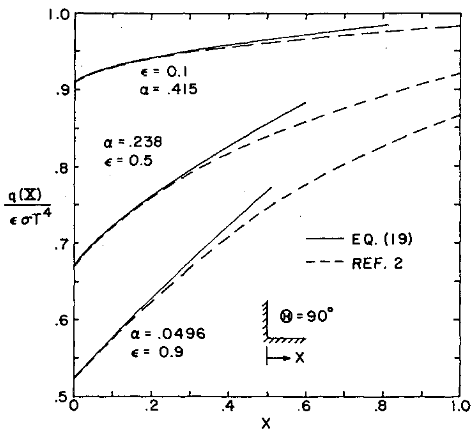

Fig. 3 Asymptotic behavior.

where

$$b \equiv (1 + \cos\theta)\csc^2\theta$$

$$Q(\alpha,\theta) \equiv \int_0^\infty u^{1-\alpha}K(1,u)du,\ 0 < \alpha < 1 \quad (10)$$

A term that varies like X was omitted in Eq. (8) because it would generate a term like $X \ln X$ on the right-hand side of Eq. (9), which is not permissible since $X \ln X$ dominates X as $X \to 0$.

Equating like powers of X gives

$$X^0:\quad \beta_0 = 1 + \lambda b \beta_0 \quad (11)$$

$$X:\quad 0 = \beta_0 + (\beta_1/\alpha) \quad (12)$$

$$X^{1-\alpha}:\quad 1 = \lambda Q(\alpha,\theta) \quad (13)$$

Solving Eq. (11) for β_0 gives $\beta_0 = \beta(0)$ as given by Eq. (6). The value of β_1 is determined from Eq. (12) as $\beta_1 = -\alpha\beta(0)$. Equation (13) determines α as a function of λ and θ.

The explicit inversion of Eq. (13) for α as a function of θ and λ is quite laborious in general. We can, however, obtain an approximate expression for small α which illustrates the behavior. Using Eq. (10), one can show

$$Q(\alpha,\theta) = \frac{1}{\alpha} + \sum_{n=0}^{\infty} \frac{(-\alpha)^n}{n!}\mu_n(\theta) \quad (14)$$

where

$$\mu_n(\theta) \equiv \int_1^\infty \frac{(\ln u)^n}{u}[u^2K(1,u) - 1]\,du + \int_0^1 u(\ln u)^n K(1,u)du \quad (15)$$

Using standard integral tables, we evaluate μ_0 as

$$\mu_0 = \ln\left[\frac{2}{1-\cos\theta}\right] + \frac{2\cos\theta - 1}{1 - \cos\theta} \quad (16)$$

For small α it follows that

$$\alpha = \lambda/(1 - \lambda\mu_0) + 0(\alpha^2) \quad (17)$$

When $\theta = \pi/2$, $Q(\alpha,\pi/2)$ can be evaluated in terms of gamma functions. We then obtain

$$\lambda = \alpha\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2} - \alpha/2)\Gamma(1 + \alpha/2)} \quad (18)$$

For small X, the heat transfer is

$$\frac{q(X)}{\epsilon\sigma T^4} \sim \frac{\sin^2(\theta/2)}{1 - \rho\cos^2(\theta/2)} + \frac{\epsilon\alpha\beta(0)}{\rho}X^{1-\alpha},\ X \to 0 \quad (19)$$

This expression shows the nonanalytic behavior when $X \to 0$. Since $0 < \alpha < 1$, the derivative of $q(X)$ is infinite at the vertex. Unless especially accounted for, this behavior causes difficulty in numerical solutions. The singular behavior becomes stronger as α increases.

Equation (19) is plotted in Fig. 3 for $\theta = \pi/2$, in which case Eq. (18) can be used to evaluate α. Also shown are the numerical results of Love and Turner,[2] which agree quite well with Eq. (19) at the vertex for this case. Moreover, Eq. (19), although valid only for small X, gives good results over a larger range than was strictly intended.

References

[1] Sparrow, E. M. et al., "Analysis, Results, and Interpretation for Radiation Between Some Simply Arranged Gray Surfaces," *Journal of Heat Transfer*, Vol. C83, 1961, pp. 207–214.

[2] Love, T. J. and Turner, W. D., "Higher-Order Approximations for Lumped System Analysis of Evacuated Enclosures," *AIAA Progress in Astronautics and Aeronautics: Thermal Design Principles of Spacecraft and Entry Bodies,* Vol. 21, edited by J.T. Bevans, Academic Press, New York, 1969, pp. 3–19.

[3] Hildebrand, F. B., *Methods of Applied Mathematics,* 2nd ed., Prentice-Hall, Englewood Cliffs, N.J., 1965, p. 266.

Effect of Implementation Delays on Errors in Linear Estimators

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Nomenclature

H = measurement matrix
K = filter gain matrix
$P(-)$ = covariance matrix of estimation errors, prior to an update
$P(+)$ = covariance matrix of estimation errors, after an update
$\mathbf{u}$ = vector of system disturbances (process noise)
$\mathbf{v}$ = vector of measurement noise
$\mathbf{x}$ = state vector, prior to a correction
$\mathbf{x}'$ = state vector, immediately after a correction
$\hat{\mathbf{x}}(-)$ = estimate of state, immediately prior to incorporating a new measurement
$\hat{\mathbf{x}}(+)$ = estimate of state, after incorporating a new measurement in the estimate but prior to correcting the state
$\hat{\mathbf{x}}'$ = estimate of state, immediately after correcting the state
$\tilde{\mathbf{x}}(-)$ = error in the state estimate, prior to incorporating a new measurement
$\tilde{\mathbf{x}}(+)$ = error in the state estimate, immediately after incorporating a measurement
$\mathbf{z}$ = measurement vector
$\Phi(t_i,t_j)$ = state transition matrix relating change in state vector over interval t_j to t_i

Subscripts

t_n = pertaining to time $t = t_n$
n = also pertaining to time $t = t_n$

Superscripts

$*$ = the version implemented in the linear estimator
T = transpose of a vector or matrix

TWO common problems facing the designer of a Kalman estimator are the finite operation times inherent in any digital computer implementation and the modeling of correlations in the measurement errors. It is frequently impractical to provide a complete treatment of correlated measure-

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